Math 3236 Statistical Theory

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X_i is a sample from $N(\mu, \sigma^2)$ σ Known

$$T = \sqrt{n}\left(\frac{\bar{X} - \mu_0}{\sigma}\right)$$

$H_0 : \mu \leq \mu_0$

$H_a : \mu > \mu_0$

δ_c : reject H_0 if $T \geq c$

$$\pi(\mu \mid \delta_c) = \mathbb{P}(T > c \mid \mu)$$

$$Z = \sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) \sim N(0,1)$$

$$T = \sqrt{n}\left(\frac{\bar{X} - \mu}{\sigma}\right) + \sqrt{n}\left(\frac{\mu - \mu_0}{\sigma}\right)$$

$$T = Z + \sqrt{N}\left(\frac{\mu - \mu_0}{\sigma}\right)$$

$$\pi(\mu \mid \delta_c) = \mathbb{P}\left(Z \geq c - \sqrt{N}\left(\frac{\mu - \mu_0}{\sigma}\right)\right)$$

The size of the test

$$\sup_{\mu \leq \mu_0} \pi(\mu \mid \delta_c) = \pi(\mu_0 \mid \delta_c)$$

$$= 1 - \Phi\left(c - \sqrt{N}\left(\frac{\mu - \mu_0}{\sigma}\right)\right)\Bigg|_{\mu = \mu_0}$$

$$= 1 - \Phi(c)$$

If I want a test of size α

$$1 - \Phi(c) = \alpha$$

$$c = \Phi^{-1}(1 - \alpha)$$

p-value of the test

If result is T for The statistic T

$$\text{p-value} = 1 - \Phi(t)$$

$$\Omega_0 = \{ \mu \leq \mu_0 \}$$

$$\Omega_1 = \{ \mu > \mu_0 \}$$

$f(\underline{x} \mid \mu)$ likelihood function for The sample

$$f(\underline{x} \mid \mu) = \left(\frac{1}{2\pi\sigma^2} \right)^{N/2} e^{- \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{2\sigma^2}}$$

How likely is H_0 :

$$\frac{\sup_{\mu \in \Omega_0} f(\underline{x} \mid \mu)}{\sup_{\mu} f(\underline{x} \mid \mu)} = \Lambda(x)$$

likelyhood ratio.

$\Lambda(x) \leq 1$

if $\Lambda(x)$ is less than 1

it means that the sup.

$f(\underline{x} \mid \mu)$ is reached for

$\mu \notin \Omega_0$.

If it is very small, it is

probably far away from

Ω_0.

δ_K the test reject H_0 if

$\Lambda(x) \leq K$ for some K.

Discussion of likelihood ratio

Test for Normal r.v.

$$f(\underline{x} \mid \mu) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\sum_i (x_i - \mu)^2 / 2\sigma^2}$$

$$\hat{\mu} = \frac{1}{N} \sum_i x_i = \overline{x}$$

$$\sup_\mu f(\underline{x} \mid \mu) = \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\sum_i (x_i - \overline{x})^2 / 2\sigma^2}$$

$$\sup_{\mu \le \mu_0} f(\underline{x} \mid \mu) = \sup_\mu f(\underline{x} \mid \mu)$$

$$\overline{x} > \mu_0$$

$$\frac{d}{d\mu} e^{-\sum_i (x_i - \mu)^2 / 2\sigma^2} = -\frac{\sum_i (x_i - \mu)}{\sigma^2} e^{-\frac{\sum_i (x_i - \mu)^2}{2\sigma^2}}$$

$\mu < \overline{x}$ $\quad$ $f(\overline{x} \mid \mu)$ is increasing

$$\sup_{\mu \le \mu_0} f(\underline{x} \mid \mu) = f(x \mid \mu_0) =$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{N/2} e^{-\sum_{i=1}^{N} (x_i - \mu_0)^2 / 2\sigma^2}$$

$$\Lambda(\underline{x}) = \frac{e^{-\sum_{i=1}^{N}(x_i-\mu_0)^2/2\sigma^2}}{e^{-\sum_{i=1}^{N}(x_i-\bar{x})^2/2\sigma^2}}$$

$$\sum_i (x_i-\bar{x})^2 = \sum_i (x_i-\mu_0+\mu_0-\bar{x})^2 =$$

$$= \sum_i (x_i-\mu_0)^2 + N(\mu_0-\bar{x})^2$$

$$+2\sum_i (x_i-\mu_0)(\mu_0-\bar{x})^2 =$$

$$= \sum_i (x_i-\mu_0)^2 - N(\mu_0-\bar{x})^2$$

$$\Lambda(\underline{x}) = e^{-N(\mu_0-\bar{x})^2/2\sigma^2}$$

$$\Lambda(\underline{x}) = \begin{cases} 1 & \bar{x} \leq \mu_0 \\ e^{-N(\bar{x}-\mu_0)^2/2\sigma^2} & \bar{x} > \mu_0 \end{cases}$$

$$e^{-N(\bar{x}-\mu_0)^2} \leq K \implies$$

$$\frac{\bar{x}-\mu_0}{\sigma} \geq \sqrt{-\log K}$$

The z Test for Normal r.v. with σ known is actually a likelihood ratio Test.

I have
H_0, H_a and a Test δ^*
such that

$$\alpha(\delta^*) = \sup_{\theta \in \Omega_0} \pi(\delta^* | \theta) \leq \alpha$$

I say That δ^* is the uniformly most powerful Test if
$\forall \delta$ such That $\alpha(\delta) \leq \alpha$
$\forall \theta$

$$\pi(\delta^* | \theta) \geq \pi(\delta | \theta)$$

$$\frac{f(\underline{x} \mid \theta_1)}{f(\underline{x} \mid \theta_2)} = F(r(\underline{X}))$$

is such a way that if $\theta_1 < \theta_2$

The F is a monotone function.

Monotone likelihood ratio (MLR)

in the statistics $T = r(\underline{X})$.

If the distribution of $\underline{X}$ has

a monotone likelihood ratio

in the statistics T

Then the Test That reject

H_0 if $T \geq c$ is UMP

Test.

The one sided Test for Normal r.v. with σ Known based on $\overline{X}$ is The UMP Test.

θ_0 $\qquad$ θ_1

$H_0 = \theta = \theta_0$ $\qquad$ $H_1 = \theta = \theta_1$

δ $\qquad$ $\alpha(\delta)$ prob rejecting H_0 when True $(\theta = \theta_0)$

$\beta(\delta)$ prob not rejecting H_0 when not True $(\theta = \theta_1)$

δ is the Test that reject H_0 if

$$\frac{f(x \mid \theta_0)}{f(x \mid \theta_1)} < K$$

Any Test δ' for which

$$\alpha(\delta') \leq \alpha(\delta)$$

Then

$$\beta(\delta') \geq \beta(\delta)$$

Moreover

$$\alpha(\delta') < \alpha(\delta)$$

Then

$$\beta(\delta') > \beta(\delta)$$

The Z statistics for Normal is The best one.